The spectrum of the Chern subring

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*Dedicated to Charles Thomas, on
the occasion of his 60th birthday*

Abstract. For certain subrings of the mod-p-cohomology of a compact Lie group, we give a description of the spectrum, analogous to Quillen's description of the spectrum of the whole cohomology ring. Subrings to which our theorem applies include the Chern subring. Corollaries include a characterization of those groups for which the Chern subring is F-isomorphic to the cohomology ring.

1. Introduction.

Let G be a compact Lie group (*e.g.*, a finite group) and let $H^*(G) = H^*(BG; \mathbb{F}_p)$ be its mod-p cohomology ring. This ring is a finitely generated graded-commutative $\mathbb{F}_p$-algebra. In [16], D. Quillen studied this ring from the viewpoint of commutative algebra. His results may be stated in terms of the prime ideal spectrum of $H^*(G)$, but the cleanest statement concerns the variety, $X_G(k)$, of algebra homomorphisms from $H^*(G)$ to an algebraically closed field k of characteristic p. The Chern subring, $\mathrm{Ch}(G) \subseteq H^*(G)$, is the subring generated by Chern classes of unitary representations of G. We give a description of $X'_G(k)$, the variety of algebra homomorphisms from $\mathrm{Ch}(G)$ to k, analogous to Quillen's description of $X_G(k)$. As corollaries of this result, we classify the minimal prime ideals of $\mathrm{Ch}(G)$, and characterize those groups G for which the natural map from $X_G(k)$ to $X'_G(k)$ is a homeomorphism.

In the case when $G = E$ is an elementary abelian p-group, *i.e.*, a direct product of copies of the cyclic group of order p, $X_E(k)$ is naturally isomorphic to $E \otimes k$, where E is viewed as a vector space over $\mathbb{F}_p$ and the tensor product is taken over $\mathbb{F}_p$. For general G, Quillen describes $X_G(k)$ as the colimit of the functor $(-) \otimes k$ over a category $\mathcal{A} = \mathcal{A}(G)$ with objects the elementary abelian subgroups of G, and morphisms those group homomorphisms that are induced by conjugation in G. Our description of $X'_G(k)$ is as the colimit of the same functor over a category $\mathcal{A}'$. This category has the same objects as Quillen's category, but a morphism in $\mathcal{A}'$ is a group homomorphism that merely preserves conjugacy in G. In other words, a group homomorphism $f\colon E_1 \to E_2$ satisfies:

$$f \in \mathcal{A} \quad \Leftrightarrow \quad \exists g \, \forall e, f(e) = g^{-1}eg,$$
$$f \in \mathcal{A}' \quad \Leftrightarrow \quad \forall e \, \exists g, f(e) = g^{-1}eg.$$

This theorem is a corollary of a more general colimit theorem, which says, roughly, that the variety for any subring of $H^*(G)$ that is both 'large' and 'natural' may be expressed as such a colimit. Other corollaries of this theorem, in the case when G is finite, include a description of the variety for the subring of $\mathrm{Ch}(G)$ generated by Chern classes of representations realizable over any subfield of $\mathbb{C}$, and a slight variation on the usual proof of

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Quillen's theorem in which transfers of Chern classes are used instead of the Evens norm map.

2. Representations and the Chern subring.

First we recall some facts concerning Chern classes [3,21]. Let $U(n)$ be the group of $n \times n$ unitary matrices, and $T(n)$ its subgroup of diagonal matrices. Then $H^*(T(n))$ is a free polynomial algebra $\mathbb{F}_p[x_1, \ldots, x_n]$ on n generators of degree two. $H^*(U(n))$ is isomorphic to $\mathbb{F}_p[c_1, \ldots, c_n]$, where c_i has degree $2i$. The map from $H^*(U(n))$ to $H^*(T(n))$ is injective and sends c_i to the ith elementary symmetric function in the x_i.

Let G be a compact Lie group. G has faithful finite-dimensional complex representations, and any finite-dimensional representation is equivalent to a unitary representation. If $\rho: G \to U(n)$ is a unitary representation of G, write $\rho: BG \to BU(n)$ for the induced map of classifying spaces, whose homotopy class depends only on the equivalence class of ρ. The ith Chern class of ρ is defined by $c_i(\rho) = \rho^*(c_i) \in H^{2i}(G)$. Define $c_.(\rho)$, the total Chern class of ρ, to be $1 + c_1(\rho) + \cdots + c_n(\rho)$. Chern classes enjoy the following properties ('Whitney sum formula' and 'naturality'), for any $\theta: G \to U(m)$ and any $f: H \to G$:

$$c_.(\rho \oplus \theta) = c_.(\rho)c_.(\theta) \qquad c_.(\rho \circ f) = f^*c_.(\rho).$$

There is a unique way to define Chern classes for virtual representations so that they continue to enjoy the above properties. Let $c'_. = 1 + c'_1 + c'_2 + \cdots$ be the unique power series in $\mathbb{F}_p[[c_1, \ldots, c_n]]$ satisfying $c'_.c_. = 1$, and define $c_i(-\rho) = \rho^*(c'_i)$. In general infinitely many of the $c_i(-\rho)$ will be non-zero, but note that they are all expressible in terms of the $c_i(\rho)$.

Definition. The Chern subring $\mathrm{Ch}(G)$ of $H^*(G)$ is the subring generated by the $c_i(\rho)$ for all i and all virtual representations ρ.

By the above remarks, the $c_i(\rho)$ as ρ ranges over the irreducible representations of G suffice to generate $\mathrm{Ch}(G)$. In the case when G is finite, it follows that $\mathrm{Ch}(G)$ is finitely generated, since G has only finitely many inequivalent irreducible representations. For general G it is also true that $\mathrm{Ch}(G)$ is finitely generated. This is a special case of the following proposition.

Proposition 2.1. *Let G be a compact Lie group, and $\rho\colon G \to U(n)$ a faithful unitary representation of G. If R is a subring of $H^*(G)$ containing each $c_i(\rho)$, then R is finitely generated.*

Proof. Venkov showed that $H^*(G)$ is finitely generated by showing that $H^*(G)$ is finite over (*i.e.*, is a finitely generated module for) $H^*(U(n))$ [22,16,5]. Now R is an $H^*(U(n))$-submodule of $H^*(G)$, so is finitely generated since $H^*(U(n))$ is Noetherian. $\square$

Remark. For G finite, the finite-generation of $H^*(G)$ is due independently to Evens and to Venkov by completely different proofs [9,22]. There is another proof (closer to Evens' than to Venkov's) in [8].

Definition. A virtual representation ρ of G is said to be p-regular if the virtual dimension of ρ is strictly positive and for every elementary abelian subgroup $E \cong (\mathbb{Z}/p)^n$ of G, the restriction to E of ρ is a direct sum of copies of the regular representation of E.

Proposition 2.2. *For each prime p, every compact Lie group G has a p-regular representation.*

Proof. G has a faithful representation in $U(n)$ for some n, and every elementary abelian subgroup of $U(n)$ is conjugate to a subgroup of $T(n)$, the torus consisting of diagonal matrices. Thus it suffices to show that $U(n)$ has a virtual representation whose restriction to $(\mathbb{Z}/p)^n \subseteq T(n)$ is the regular representation.

Recall that the representation ring $R(T(n))$ of $T(n)$ is isomorphic to the Laurent polynomial ring $\mathbb{Z}[\tau_1, \tau_1^{-1}, \ldots, \tau_n, \tau_n^{-1}]$, where τ_i is the 1-dimensional representation

$$\tau_i\colon \mathrm{diag}(\xi_1, \ldots, \xi_n) \mapsto \xi_i.$$

$R(U(n))$ maps injectively to $R(T(n))$ with image the subring $\mathbb{Z}[\sigma_1, \sigma_2, \ldots, \sigma_n, \sigma_n^{-1}]$, where σ_i is the ith elementary symmetric function in the τ_j.

The polynomial

$$P = \prod_{i=1}^{n}(1 + \tau_i + \cdots + \tau_i^{p-1})$$

is a symmetric polynomial in the τ_j, and so is expressible in terms of $\sigma_1, \ldots, \sigma_n$. The corresponding ($p^n$-dimensional) representation of $U(n)$ restricts to $(\mathbb{Z}/p)^n \subseteq T(n)$ as the regular representation. $\square$

Remark. For G finite, the regular representation of G is of course p-regular.

Using Quillen's result that we state as Theorem 5.1, it may be shown that $H^*(G)$ is finite over the subring generated by the Chern classes of any p-regular representation. For genuine (as opposed to virtual) representations, this can be deduced from Venkov's result: If $\rho\colon G \to U(n)$ is a p-regular representation of G, the kernel of ρ contains no elements of order p, and is therefore a finite group of order coprime to p. It follows that $H^*(\rho(G)) \cong H^*(G)$ (consider the spectral sequence for the extension $\ker(\rho) \to G \to \rho(G)$), and hence $H^*(G)$ is finite over the image of ρ^*. When $p = 2$, the representation constructed in the proof of Proposition 2.2 is a genuine representation. David Kirby has shown us an argument to prove that $U(n)$ has a p-regular genuine representation if and only if either $p = 2$, or $n = 1$, or $(p, n) = (3, 2)$.

3. Varieties for cohomology.

Let k be an algebraically closed field of characteristic p, and let R be a finitely generated commutative $\mathbb{F}_p$-algebra. Define $V_R(k)$, the variety for R, to be the set of ring homomorphisms from R to k, with the Zariski topology, *i.e.*, the smallest topology in which the set

$$F_I = \{\phi \colon R \to k \mid \ker(\phi) \supseteq I\}$$

is closed for each ideal I of R. A ring homomorphism $f \colon R \to S$ gives rise to a continuous map $f^* \colon V_S(k) \to V_R(k)$. If S is finite over $f(R)$ (*i.e.*, S is a finitely generated $f(R)$-module) then f^* is a closed mapping and has finite fibres, by the 'going up' or 'lying over' theorem [4,6]. If S is finite over $f(R)$ and $\ker(f)$ is nilpotent, then f^* is surjective.

There is a continuous map from $V_R(k)$ to $\mathrm{Spec}(R)$, the prime ideal spectrum of R, that sends the map $\phi \colon R \to k$ to the ideal $\ker(\phi)$. If the transcendence degree of k over $\mathbb{F}_p$ is sufficiently large (as large as a generating set for R will suffice), then this map is surjective. Thus information about $V_R(k)$ gives rise to information about $\mathrm{Spec}(R)$.

We shall also require the following:

Proposition 3.1. (a) *Let S be a subring of R containing R^p, the subring of pth powers of elements of R. Then the natural map from $V_R(k)$ to $V_S(k)$ is a homeomorphism.*
(b) *Let a finite group G act on R, with fixed point subring $S = R^G$. Then the natural map $V_R(k) \to V_S(k)$ induces a homeomorphism $V_R(k)/G \to V_S(k)$.*

Proof. Each of these claims may be proved by showing that R is finite over S, and deducing that the map given is continuous, closed, and a bijection. See for example [4,6,16]. $\square$

For $p = 2$, the ring $H^*(G)$ is commutative. For p an odd prime, elements of $H^*(G)$ of odd degree are nilpotent, so although $H^*(G)$ is not commutative, the quotient of $H^*(G)$ by its radical, $h^*(G) = H^*(G)/\sqrt{0}$, is commutative. Any homomorphism from $H^*(G)$ to k factors through $h^*(G)$. Define $X_G(k)$ to be the variety $V_R(k)$ for $R = h^*(G)$. By the above remark, points of $X_G(k)$ may be viewed as homomorphisms from $H^*(G)$ to k. Let S be the subring of elements of $H^*(G)$ of even degree. A homomorphism $\phi \colon S \to k$ extends uniquely to a homomorphism from $H^*(G)$ to k (if x is in odd degree, then either p is odd, and $x^2 = 0$, or $p = \mathrm{char}(k) = 2$, so in either case $\phi(x)$ is the unique square root of $\phi(x^2)$). It follows that the natural map $X_G(k) \to V_S(k)$ is a homeomorphism, since it is a closed, continuous bijection. Hence $X_G(k)$ could equally be defined in terms of S.

Note that a group homomorphism $f \colon H \to G$ induces a map $f_* \colon X_H(k) \to X_G(k)$. We write ι_H^G for f_* in the case when f is the inclusion of a subgroup H in G. A theorem of Evens and Venkov [9,22] states that in this case $H^*(H)$ is finite over $H^*(G)$. (To deduce this from the result quoted in the proof of Proposition 2.1, note that a faithful representation of G restricts to a faithful representation of H.) It follows that ι_H^G is closed and has finite fibres.

Define $X_G'(k)$ to be $V_{\mathrm{Ch}(G)}(k)$. By Proposition 2.1 the natural map from $X_G(k)$ to $X_G'(k)$ is surjective, closed, and has finite fibres.

Proposition 3.2. *Let ρ be a representation of G, and let $R(n)$ be the subring of $H^*(G)$ generated by the Chern classes of $n\rho = \rho^{\oplus n} = \rho \oplus \cdots \oplus \rho$. Then the natural map $V_{R(1)}(k) \to V_{R(n)}(k)$ is a homeomorphism.*

Proof. If p does not divide n, then

$$c_i(n\rho) = nc_i(\rho) + P(i, n),$$

for some expression $P(i, n)$ in the $c_j(\rho)$ for $j < i$. So in this case $R(n) = R(1)$. On the other hand, if $n = pm$ then

$$c.(n\rho) = c.(m\rho)^p = 1 + c_1(m\rho)^p + c_2(m\rho)^p + \cdots,$$

so in this case $R(n) = R(m)^p$, the subring of pth powers of elements of $R(m)$, and the map $V_R(k) \to V_{R^p}(k)$ is a homeomorphism. $\square$

The methods that we shall use to study the Chern subring apply equally to the Stiefel-Whitney subring, defined analogously, in the case when $p = 2$. (For information concerning Stiefel-Whitney classes see [21]). As an alternative, the following proposition may be applied.

Proposition 3.3. *Let $p = 2$ and let S be the subring of $H^*(G)$ generated by Stiefel-Whitney classes of real representations of G. Then*

$$S^2 \subseteq \mathrm{Ch}(G) \subseteq S,$$

and the natural map from $V_S(k)$ to $X'_G(k)$ is a homeomorphism.

Proof. If θ is an n-dimensional real representation of G, then $\theta^{\mathbb{C}}$, the complexification of θ, is an n-dimensional complex representation of G with $c_i(\theta^{\mathbb{C}}) = w_i^2(\theta)$. Conversely, if ψ is an n-dimensional complex representation of G and $\psi_{\mathbb{R}}$ is the same representation viewed as a $2n$-dimensional real representation, then $w_i(\psi_{\mathbb{R}}) = 0$ for i odd and $w_{2i}(\psi_{\mathbb{R}}) = c_i(\psi)$. This proves the claimed inclusions. The claimed homeomorphism follows from Proposition 3.1(a). $\square$

4. Examples.

In this section we discuss the case of an elementary abelian p-group, and also give an example to show that the map $X_G(k) \to X'_G(k)$ is not always a homeomorphism. This example was the starting point for the work of this paper.

Let E be an elementary abelian p-group of rank n, $E \cong (\mathbb{Z}/p)^n$. Then E may be viewed as a vector space over $\mathbb{F}_p$. Write E^* for $\mathrm{Hom}(E, \mathbb{F}_p)$. There is a natural isomorphism $E^* \cong H^1(E)$. For $p = 2$, $H^*(E)$ is isomorphic to the symmetric algebra on $H^1(E)$, or equivalently, the ring of polynomial functions on E viewed as a vector space:

$$H^*(E) \cong S(E^*) \cong \mathbb{F}_p[E].$$

For $p > 2$, the Bockstein $\beta \colon H^1(E) \to H^2(E)$ is injective, and $H^*(E)$ is isomorphic to the tensor product of the exterior algebra on $H^1(E)$ tensored with the symmetric algebra on $B = \beta(H^1(E))$:

$$H^*(E) \cong \Lambda(E^*) \otimes S(B) \cong \Lambda(E^*) \otimes \mathbb{F}_p[E].$$

In any case, $h^*(E)$ is naturally isomorphic to $\mathbb{F}_p[E]$, generated in degree one for $p = 2$ and in degree two for odd p. It follows that $X_E(k)$ is naturally isomorphic to $E \otimes k$, where E is viewed as a vector space over $\mathbb{F}_p$ and the tensor product is over $\mathbb{F}_p$, so that $E \otimes k \cong k^n$.

Irreducible representations of E are 1-dimensional, and the map $\rho \mapsto c_1(\rho)$ is a natural bijection between the set of irreducible representations of E and $B = \beta(H^1(E))$. (When $p = 2$, $\beta = \mathrm{Sq}^1$, and so $\beta(x) = x^2$.) The Chern subring $\mathrm{Ch}(E)$ of $H^*(E)$ is the subalgebra of $H^*(E)$ generated by B. For $p > 2$ this subring maps onto $h^*(E)$, and for $p = 2$ it maps onto $h^*(E)^2$, the subring of squares of elements of $h^*(E)$. In any case, the map from $X_E(k)$ to $X'_E(k)$ is a homeomorphism.

Proposition 4.1. *Let ρ be a direct sum of copies of the regular representation of E, and let R be the subring of $H^*(E)$ generated by the Chern classes of ρ. Then the natural map from $X_E(k)$ to $V_R(k)$ factors through a homeomorphism*

$$k^n/\mathrm{GL}_n(\mathbb{F}_p) \cong X_E(k)/\mathrm{GL}(E) \to V_R(k).$$

Proof. By Proposition 3.2 it suffices to consider the case when ρ is the regular representation. Identify $\mathrm{Ch}(E)$ with $\mathbb{F}_p[E]$, generated in degree two. The total Chern class of ρ is

$$c.(\rho) = \prod_{x \in E^*} (1 + x).$$

This is invariant under the full automorphism group, $\mathrm{GL}(E)$, of E. By a theorem of Dickson, the only $i > 0$ for which $c_i(\rho)$ is non-zero are $i = p^n - p^j$, where $0 \le j < n = \dim_{\mathbb{F}_p}(E)$. Moreover, these $c_i(\rho)$ freely generate a polynomial subring of $\mathbb{F}_p[E]$, and this is the complete ring of $\mathrm{GL}(E)$-invariants in $\mathbb{F}_p[E]$ [6,23]. The claim follows by part (b) of Proposition 3.1. $\square$

Remark. Let A be a non-identity element of $\mathrm{GL}_n(\mathbb{F}_p)$, and let v be an element of k^n fixed by A. Then v is in the kernel of $I - A$, a non-zero matrix with entries in $\mathbb{F}_p$, and so v lies in a proper subspace of k^n defined over $\mathbb{F}_p$ (*i.e.*, a subspace of the form $V \otimes k$ for some proper $\mathbb{F}_p$-subspace V of $\mathbb{F}_p^n$). It follows that $\mathrm{GL}_n(\mathbb{F}_p)$ acts freely on the complement of all such subspaces. For an elementary abelian group E, let

$$X_E^+(k) = X_E(k) \setminus \bigcup_{F < E} \iota_F^E(X_F(k)).$$

By the above argument, $\mathrm{GL}(E)$ acts freely on $X_E^+(k)$.

Example. Let $q = p^n$ for some $n \ge 2$. Let G be the affine transformation group of the line over $\mathbb{F}_q$. Then G is expressible as an extension with kernel $E = (\mathbb{F}_q, +)$, an elementary abelian p-group of rank n, and quotient $Q = \mathrm{GL}_1(\mathbb{F}_q)$, cyclic of order $q - 1$. The conjugation action of Q on E is transitive on non-identity elements of E. One example of such a group is the alternating group A_4 ($p = 2$, $n = 2$). An easy transfer argument shows that $H^*(G)$ maps isomorphically to the ring of invariants, $H^*(E)^Q$, and it follows from Proposition 3.1 that $X_G(k)$ is homeomorphic to $X_E(k)/Q = k^n/Q$, where $Q = \mathrm{GL}_1(\mathbb{F}_q) \le \mathrm{GL}_n(\mathbb{F}_p) \le \mathrm{GL}_n(k)$.

It is easy to check that G has exactly q distinct irreducible representations. All but one of these are 1-dimensional and restrict to E as the trivial representation. The other one is $(q-1)$-dimensional and restricts to E as the regular representation minus the trivial representation. Hence by Proposition 4.1, $X'_G(k)$ is homeomorphic to $X_E(k)/\mathrm{GL}(E) = k^n/\mathrm{GL}_n(\mathbb{F}_p)$. Thus the map from $X_G(k)$ to $X'_G(k)$ is not a homeomorphism.

5. Quillen's colimit theorem.

In [16], Quillen showed that for general G, $X_G(k)$ is determined by the elementary abelian subgroups of G. Roughly speaking, he showed that $X_G(k)$ is equal to the union of the images of the $X_E(k)$, where E ranges over the elementary abelian subgroups of G, and that as little identification takes place between the points of the $X_E(k)$ as is consistent with the fact that inner automorphisms of G act trivially on $H^*(G)$. More precisely, let $f\colon E_1 \to E_2$ be a homomorphism between elementary abelian subgroups of G that is induced by an inner automorphism of G. Then the following diagram commutes.

$$
\begin{array}{ccc}
H^*(G) & \xleftarrow{\;\mathrm{Id}\;} & H^*(G) \\
\downarrow{\scriptstyle\mathrm{Res}} & & \downarrow{\scriptstyle\mathrm{Res}} \\
H^*(E_1) & \xleftarrow{\;f^*\;} & H^*(E_2)
\end{array}
$$

Consequently the following diagram commutes.

$$
\begin{array}{ccc}
X_G(k) & \xrightarrow{\;\mathrm{Id}\;} & X_G(k) \\
\uparrow{\scriptstyle\iota} & & \uparrow{\scriptstyle\iota} \\
X_{E_1}(k) & \xrightarrow{\;f_*\;} & X_{E_2}(k)
\end{array}
$$

This fact motivates the following definition.

Definition. The Quillen category $\mathcal{A}$ for a compact Lie group G and a prime p is the category whose objects are the elementary abelian p-subgroups of G, with morphisms from E_1 to E_2 being those group homomorphisms that are induced by conjugation in G. Any such group homomorphism is of course injective.

In general G will have infinitely many elementary abelian p-subgroups. These subgroups form finitely many conjugacy classes though ([16], lemma 6.3). Thus although the Quillen category for G is infinite in general, it contains only finitely many isomorphism types of object (or is 'skeletally finite').

The morphisms $f\colon E_1 \to E_2$ in the Quillen category are precisely the maps for which the diagram above commutes. It follows that the natural map

$$
\coprod_{\substack{E \leq G \\ E \text{ el. ab.}}} X_E(k) \longrightarrow X_G(k)
$$

factors through a map $\alpha\colon \mathrm{colim}_{\mathcal{A}} X_E(k) \to X_G(k)$.

Theorem 5.1. *(Quillen [16]) The map $\alpha\colon \mathrm{colim}_{\mathcal{A}} X_E(k) \to X_G(k)$ is a homeomorphism.*

The map α is continuous and is closed because $\mathcal{A}$ is skeletally finite. Thus the main content of the theorem is that α is a bijection. We shall use only half of this theorem, the statement that α is surjective, in our main theorem. The surjectivity of α is equivalent to the statement 'an element of $H^*(G)$ is nilpotent if and only if its image in each $H^*(E)$ is nilpotent'.

6. A new colimit theorem.

Motivated by the Quillen category, we define:

Definition. A category of elementary abelian subgroups of G is a category whose objects are (all of) the elementary abelian p-subgroups of G, and whose morphisms from E_1 to E_2 are injective group homomorphisms.

The Quillen category, $\mathcal{A}(G)$, is of course a category of elementary abelian subgroups of G. Another example is the category $\mathcal{C}_{\mathrm{reg}}(G)$, with the morphism set $\mathcal{C}_{\mathrm{reg}}(E_1, E_2)$ equal to the set of all 1-1 group homomorphisms from E_1 to E_2. Any category of elementary abelian subgroups of G is a subcategory of $\mathcal{C}_{\mathrm{reg}}(G)$.

Any subring R of $H^*(G)$ gives rise to a category $\mathcal{C}(R)$ of elementary abelian subgroups of G, where $f\colon E_1 \to E_2$ is a morphism in $\mathcal{C}(R)$ if and only if

$$
\begin{array}{ccc}
R & \xleftarrow{\ \mathrm{Id}\ } & R \\
\downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} \\
h^*(E_1) & \xleftarrow{\ f^*\ } & h^*(E_2)
\end{array}
$$

commutes. Note that we use $h^*(E_i)$, the cohomology ring modulo its radical, rather than $H^*(E_i)$. Note that if f is an isomorphism of groups and is a morphism in $\mathcal{C}(R)$, f^{-1} is also in $\mathcal{C}(R)$. Each $\mathcal{C}(R)$ contains the Quillen category, and hence is skeletally finite. By the argument given in Section 5, the map $\alpha\colon \mathrm{colim}_{\mathcal{A}} X_E(k) \to X_G(k)$ induces a map $\gamma = \gamma(R)\colon \mathrm{colim}_{\mathcal{C}(R)} X_E(k) \to V_R(k)$.

Definition. Say that a subring of $H^*(G)$ is large if it contains the Chern classes of some p-regular representation of G. Say that a subring of $H^*(G)$ is natural if it is generated by homogeneous elements and is closed under the action of the Steenrod algebra.

The new colimit theorem of the title of this section is:

Theorem 6.1. *Let G be a compact Lie group, and let R be a subring of $H^*(G)$ that is both large and natural. Then the map*

$$
\gamma\colon \mathrm{colim}_{\mathcal{C}(R)} X_E(k) \to V_R(k)
$$

is a homeomorphism.

It is possible that this theorem could be proved using more general colimit theorems due to S. P. Lam, to D. Rector, and to H.-W. Henn, J. Lannes and L. Schwartz [14,18,12]. These theorems say, roughly speaking, that the variety for any Noetherian algebra over

the Steenrod algebra should be expressible as a similar sort of colimit. Even with these theorems, Quillen's description of $X_G(k)$ would still be needed to identify the categories that arise with categories of elementary abelian subgroups of G. The proof given below is more elementary, in that it relies on no work that is more recent than that of Quillen.

The proof of the theorem uses the following lemma.

Lemma 6.2. *Let S be the subring of $H^*(G)$ generated by the Chern classes of some p-regular representation of G. Then $\mathcal{C}(S)$ is equal to the category $\mathcal{C}_{\mathrm{reg}}$ defined above, and the map $\gamma(S)$ is a homeomorphism.*

Proof. Let F be a maximal elementary abelian subgroup of G. Note that the natural map from $X_F(k)$ (mapping the category with one object and one morphism to $\mathcal{C}_{\mathrm{reg}}$) induces a homeomorphism
$$X_F(k)/\mathrm{GL}(F) \cong \mathrm{colim}_{\mathcal{C}_{\mathrm{reg}}} X_E(k).$$
By Proposition 4.1, the image of $\iota_F^G \colon X_F(k) \to V_S(k)$ is homeomorphic to $X_F(k)/\mathrm{GL}(F)$. If E is any elementary abelian subgroup of G and $f \colon E \hookrightarrow F$ is any injective group homomorphism, then $\mathrm{Res}_E^G(\rho)$ and $f^*\mathrm{Res}_F^G(\rho)$ are equal to a sum of (the same number of) copies of the regular representation of E. Hence $\mathcal{C}(S)$ is equal to $\mathcal{C}_{\mathrm{reg}}$, and $\mathrm{Im}(\gamma) = \mathrm{Im}(\iota_F^G)$. It follows that γ is a homeomorphism onto its image. Finally, by Theorem 5.1, this image is the whole of $V_S(k)$. $\square$

Proof of the theorem. Since R is large, it contains a subring S satisfying the conditions of Lemma 6.2. Let E, F be two elementary abelian subgroups of G, suppose that the rank of E is less than or equal to that of F, and suppose that $\phi \in X_E(k)$ and $\psi \in X_F(k)$ define the same point of $V_R(k)$. *A fortiori* ϕ and ψ define the same point of $V_S(k)$, and so by Lemma 6.2 there is an injective group homomorphism $f \colon E \hookrightarrow F$ such that $\psi = \phi \circ f^* = f_*(\phi)$. It suffices to show that such an f is in $\mathcal{C}(R)$.

For any such $f \colon E \hookrightarrow F$, let $\mathcal{S}$ be the set of subgroups of E such that f restricted to E is a morphism in $\mathcal{C}(R)$:
$$\mathcal{S} = \{E' \leq E \mid (f|_{E'} \colon E' \to F) \in \mathcal{C}(R)\},$$
and define a subset $X(f)$ of $X_E(k)$ by
$$X(f) = \{\phi \in X_E(k) \mid f_*(\phi) \circ \mathrm{Res}_{E'}^G|_R = \psi \circ \mathrm{Res}_F^G|_R\}.$$
From the definitions,
$$X(f) \supseteq \bigcup_{E' \in \mathcal{S}} \iota_{E'}^E X_{E'}(k),$$
and it suffices to show that equality holds. Note that a subgroup $E' \leq E$ is in $\mathcal{S}$ if and only if $\iota X_{E'}(k)$ is a subset of $X(f)$. Hence it suffices to show that $X(f)$ is equal to some union of sets of the form $\iota X_{E'}(k)$. Now let $I(f)$ be the ideal of $H^*(E)$ generated by all elements of the form $\mathrm{Res}_E^G(r) - f^*\mathrm{Res}_F^G(r)$, where $r \in R$. The subvariety of $X_E(k)$ defined by $I(f)$ is the set $X(f)$ defined above. Since R is natural (in the sense defined above the statement), the ideal $I(f)$ is homogeneous and closed under the action of the Steenrod algebra. But by a theorem of Serre [19,16], the variety corresponding to any such ideal of $H^*(E)$ has the required form. $\square$

Minimal prime ideals of a commutative $\mathbb{F}_p$-algebra R correspond to irreducible components of $V_R(k)$. Hence one obtains:

Corollary 6.3. *Let R be a large, natural subring of $H^*(G)$. The minimal prime ideals of R are in bijective correspondence with the isomorphism types of maximal objects in $\mathcal{C}(R)$.*

An object of a category is called maximal if every map from it is an isomorphism. An isomorphism class of maximal objects in the Quillen category is a conjugacy class of maximal elementary abelian subgroups of G.

Corollary 6.4. *Let R and S be large natural subrings of $H^*(G)$, and suppose that R is a subring of S. The natural map $V_S(k) \to V_R(k)$ is a homeomorphism if and only if the categories $\mathcal{C}(R)$ and $\mathcal{C}(S)$ are equal.*

Proof. A direct proof could be given at this stage, but it is easier to apply Proposition 9.1, which implies that no subcategory of $\mathcal{C}_{\mathrm{reg}}$ that strictly contains $\mathcal{C}(S)$ gives rise to the same variety. $\square$

7. Applications to rings of Chern classes.

We start by defining some categories of elementary abelian subgroups of G.

Definition. Define categories of elementary abelian subgroups of G, $\mathcal{A}'$, $\mathcal{A}'_{\mathbb{R}}$, $\mathcal{A}'_{\mathrm{P}}$, and for each d dividing $p-1$, $\mathcal{A}_d$, by stipulating that an injective homomorphism $f \colon E \to F$ is in:

    $\mathcal{A}'$ if $\forall e$, $f(e)$ is conjugate (in G) to e;
    $\mathcal{A}'_{\mathbb{R}}$ if $\forall e$, $f(e)$ is conjugate to e or to e^{-1};
    $\mathcal{A}'_{\mathrm{P}}$ if $\forall e$, the subgroups $\langle e \rangle$ and $\langle f(e) \rangle$ are conjugate;
    $\mathcal{A}'_d$ if $\forall e$, $f(e)$ is conjugate to $\xi(e)$ for some ξ in the order d subgroup of $\mathrm{Aut}(\langle e \rangle)$.

Note that for $p = 2$, $\mathcal{A}' = \mathcal{A}'_{\mathbb{R}} = \mathcal{A}'_{\mathrm{P}}$, and for odd p, $\mathcal{A}'_{\mathbb{R}} = \mathcal{A}'_2$, and $\mathcal{A}'_P = \mathcal{A}'_{p-1}$. Note also that the difference between $\mathcal{A}'$ and the Quillen category $\mathcal{A}$ is the difference between '$\forall e \,\exists g\, f(e) = g^{-1}eg$' and '$\exists g \,\forall e\, f(e) = g^{-1}eg$'. The reason for introducing these categories is the following proposition.

Proposition 7.1. *Let K be a subfield of $\mathbb{C}$ and let $|K(\zeta_p) \colon K| = d$, where ζ_p is a primitive pth root of 1. Let G be a compact Lie group, and in cases (c) and (d) suppose that G is finite. Let R be the subring of $H^*(G)$ generated by Chern classes of:*
    (a) All representations of G;
    (b) Representations of G realisable over the reals;
    (c) Permutation representations of G;
    (d) Representations of G realisable over K.
    In each case, the variety $V_R(k)$ is homeomorphic to $\mathrm{colim}_{\mathcal{C}(R)} X_E(k)$. The category $\mathcal{C}(R)$ is:

$$\text{(a) } \mathcal{A}', \quad \text{(b) } \mathcal{A}'_{\mathbb{R}}, \quad \text{(c) } \mathcal{A}'_{\mathrm{P}}, \quad \text{(d) } \mathcal{A}'_d.$$

Proof. In each case, the morphisms in the category given are precisely those group homomorphisms for which $\chi(e) = \chi(f(e))$ for all characters χ coming from representations of the given type. (See [20] Chapter 12 for case (d), and for case (c) note that if e, e' are elements of G of order p, then the permutation action of e on $G/\langle e' \rangle$ has a fixed point if and only if $\langle e \rangle$ is conjugate to $\langle e' \rangle$.) The proposition therefore follows from the lemma below. $\square$

Lemma 7.2. *Let A be an additive subgroup of the representation ring of G, generated by genuine representations, and containing a p-regular representation. Let R be the subring of $H^*(G)$ generated by the Chern classes of all elements of A. Then R is large and natural, and hence by Theorem 6.1*

$$\gamma \colon \operatorname{colim}_{\mathcal{C}(R)} X_E(k) \to V_R(k)$$

is a homeomorphism. Furthermore, $f \colon E \hookrightarrow F$ is a morphism in $\mathcal{C}(R)$ if and only if for all $e \in E$, and all characters χ of elements of A, $\chi(e) = \chi(f(e))$.

Proof. First, suppose that A is generated by a single representation ρ. The image of $\rho^* \colon H^*(U(n)) \to H^*(G)$ is natural since $H^*(U(n))$ is graded and acted upon by the Steenrod algebra. The general case follows from the Cartan formula. By hypothesis, R is large. The claimed homeomorphism follows from Theorem 6.1, and it only remains to describe $\mathcal{C}(R)$.

A representation is determined up to equivalence by its character. Hence for any f as in the statement and ρ a generator of A, $f^* \operatorname{Res}_F^G c.(\rho) = \operatorname{Res}_E^G c.(\rho)$, and so any such f is in $\mathcal{C}(R)$. For the converse, note that since $\operatorname{Ch}(E)$ is a unique factorization domain, a representation of E is determined up to equivalence by its dimension and its total Chern class. Thus if $f \colon E \hookrightarrow F$ is a homomorphism for which there exists e and χ with $\chi(f(e)) \neq \chi(e)$, there exists i and ρ, a generator of A, such that $\operatorname{Res}_E^G(c_i(\rho)) - f^* \operatorname{Res}_F^G(c_i(\rho)) \neq 0$. Hence f is not in $\mathcal{C}(R)$. $\square$

Quillen's description of $X_G(k)$ (Theorem 5.1, and Theorem 8.1), Corollary 6.4 and Proposition 7.1 together yield:

Corollary 7.3. *The natural map $X_G(k) \to X'_G(k)$ is a homeomorphism if and only if the categories $\mathcal{A}(G)$ and $\mathcal{A}'(G)$ are equal.* $\square$

Example. (A p-group G for which the map $X_G(k) \to X'_G(k)$ is not a homeomorphism.) Let E be the additive group $\mathbb{F}_p^n$ for some $n > 2$, and let $U \leq \operatorname{GL}(E) = \operatorname{GL}_n(\mathbb{F}_p)$ be the Sylow p-subgroup of $\operatorname{GL}(E)$ consisting of upper triangular matrices. Let Q be the subgroup of U consisting of all matrices $(a_{i,j})$ that are constant along diagonals, i.e., $a_{i,j} = a_{i+1,j+1}$ whenever $1 \leq i < m$ and $1 \leq j < m$. Finally, let G be the split extension with kernel E and quotient Q. The group E is a maximal elementary abelian subgroup of G. Easy matrix calculations show that the orbits of the action of Q on elements of E are equal to the orbits of the action of U, and that any element of $\operatorname{GL}(E)$ that preserves the U-orbits in E is fact an element of U. It follows that the image of $X_E(k)$ in $X'_G(k)$ is $X_E(k)/U$, whereas the image of $X_E(k)$ in $X_G(k)$ is of course $X_E(k)/Q$. Thus G is a p-group such that the fibres of the map $X_G(k) \to X'_G(k)$ above general points of one irreducible component have order $|U:Q| = p^{(n-1)(n-2)/2}$.

Example. (A group G for which $X'_G(k)$ has fewer irreducible components than $X_G(k)$.) Let G be $\operatorname{GL}_3(\mathbb{F}_p)$. There are two distinct Jordan forms for elements of order p in G (resp. one if $p = 2$), and hence G has two conjugacy classes (resp. one conjugacy class if $p = 2$) of elements of order p. All maximal elementary abelian subgroups of G have rank two. The

subgroups

$$E_1 = \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

are maximal elementary abelian subgroups, and are not conjugate, although every non-identity element of E_1 is conjugate to every non-identity element of E_2. It follows that the images of $X_{E_1}(k)$ and $X_{E_2}(k)$ in $X_G(k)$ are distinct irreducible components of $X_G(k)$, whereas their images in $X'_G(k)$ are equal.

8. Transfers of Chern classes.

Throughout this section, G shall be a finite group. Following Moselle [15], we consider the 'Mackey closure' of $\mathrm{Ch}(G)$, or in other words the smallest natural subring of $H^*(G)$ that contains $\mathrm{Ch}(G)$ and is closed under transfers. More formally, we make the following:

Definition. Let G be a finite group. Define $\overline{\mathrm{Ch}}(G)$ recursively as the subring of $H^*(G)$ generated by $\mathrm{Ch}(G)$ and the image of $\overline{\mathrm{Ch}}(H)$ under the transfer Cor_H^G for each proper subgroup $H < G$.

We shall prove:

Theorem 8.1. Let G be a finite group, with Quillen category $\mathcal{A}$, and let $R = \overline{\mathrm{Ch}}(G)$. The map α induces a homeomorphism $\alpha \colon \mathrm{colim}_{\mathcal{A}} X_E(k) \to V_R(k)$.

We do not use the injectivity of Quillen's map in proving Theorem 8.1, so two immediate corollaries of Theorem 8.1 are:

Corollary 8.2. (Quillen) For a finite group G, the map $\alpha \colon \mathrm{colim}_{\mathcal{A}} X_E(k) \to X_G(k)$ is injective. $\square$

Corollary 8.3. For a finite group G, the inclusion of $\overline{\mathrm{Ch}}(G)$ in $H^*(G)$ induces a homeomorphism of varieties. $\square$

Proof of the theorem. The transfer map Cor_H^G commutes with the action of the Steenrod algebra, by either a topological argument [1] or an algebraic one [10]. It follows that $R = \overline{\mathrm{Ch}}(G)$ is a large natural subring of $H^*(G)$. By Theorem 6.1, it suffices to show that $\mathcal{C}(R) = \mathcal{A}$.

First we show that if E, F are elementary abelian subgroups of G such that E is not conjugate to a subgroup of F, then there is no map in $\mathcal{C}(R)$ from E to F. Since any map is the composite of an isomorphism followed by the inclusion of a subgroup, it suffices to consider the case when E and F have the same rank. (Note that if f is any map in $\mathcal{C}(R)$ that is an isomorphism of elementary abelian groups, then the inverse of f is also in $\mathcal{C}(R)$, so f is an isomorphism in $\mathcal{C}(R)$.)

Let $N = N_G(F)$ be the normalizer of F in G, let $\theta = \mathbb{C}F - 1$ be the $(|F| - 1)$-dimensional representation of F given by the difference of the regular representation and the trivial representation, and let $\rho = \mathrm{Ind}_F^N(\theta)$ be the induced representation of N. Equivalently, ρ is the regular representation of N minus the permutation representation on the cosets N/F of F. Note that ρ is a genuine representation of N of dimension $|N| - |N{:}F|$.

Now let F' be any elementary abelian subgroup of N. The regular representation of N restricts to F' as a sum of $|N:F'|$ copies of the regular representation of F'. The number of orbits of F' on the cosets N/F, or equivalently the number of trivial F'-summands of the permutation module $\mathbb{C}N/F$, is equal to $|N:F'F|$. It follows that $\mathrm{Res}_{F'}^N(\rho)$, the restriction to F' of ρ, contains the trivial F'-module as a direct summand if and only if $F'F \neq F'$, *i.e.*, if and only if F is not a subgroup of F'.

Let n be the dimension of ρ, so that $n = |N|(1 - 1/|F|)$, and note that since the restriction to F of ρ does not contain the trivial representation, $\mathrm{Res}_F^N(c_n(\rho))$ is non-zero. Let $x_F = \mathrm{Cor}_N^G(c_n(\rho))$. For E an elementary abelian subgroup of G of the same rank as F, the Mackey formula affords a calculation of $\mathrm{Res}_E^G(x_F)$:

$$\mathrm{Res}_E^G(x_F) = \sum_{EgN} \mathrm{Cor}_{E \cap gNg^{-1}}^E c_g^* \mathrm{Res}_{g^{-1}Eg \cap N}^N(c_n(\rho)),$$

where for any subgroup K of G, c_g is the homomorphism $k \mapsto g^{-1}kg$, and the sum is over a set of double coset representatives for $E \backslash G/N$. The restriction map from E to any subgroup is surjective in cohomology, and $\mathrm{Cor}_{E'}^E \mathrm{Res}_{E'}^E$ is equal to multiplication by $|E:E'|$. Hence the transfer $\mathrm{Cor}_{E'}^E$ is zero for any proper subgroup E' of E. Thus the only non-zero contributions to the above sum come from terms in which $g^{-1}Eg \leq N$. On the other hand, we know that the restriction of $c_n(\rho)$ to an elementary abelian subgroup of N is non-zero if and only if that subgroup contains F. Since we are assuming that E has the same rank as F, it follows that the only non-zero terms will come from g such that $F = g^{-1}Eg$. If $F = g^{-1}Eg = h^{-1}Eh$, then $g^{-1}hFh^{-1}g = F$, so $g^{-1}h \in N$, and so $EgN = EhN$. It follows that, for E and F of the same rank, $\mathrm{Res}_E^G(x_F) = c_g^*(c_n(\rho))$ for any g such that $g^{-1}Eg = F$, and is zero if there is no such g, *i.e.*, if E and F are not conjugate in G. Since $\mathrm{Res}_F^N c_n(\rho)$ is non-zero in $h^*(F)$, it follows that when E and F have the same rank, there are morphisms from E to F in $\mathcal{C}(R)$ only if E is conjugate to F.

It remains to show that the automorphisms of F in the category $\mathcal{C}(R)$ are precisely the maps induced by conjugation in G. Let $C = C_G(F)$ be the centralizer of F in G, and suppose that $|C:F| = p^m r$, for some r coprime to p. For any representation λ of F, $\mathrm{Res}_F^C \mathrm{Ind}_F^C(\lambda) = p^m r \lambda$. It follows that the image of $\mathrm{Ch}(C)$ in $\mathrm{Ch}(F)$ contains the subring of p^mth powers. In $\mathbb{F}_p[F] = \mathrm{Ch}(F)$, there is a homogeneous element y_1 such that the $\mathbb{F}_p\mathrm{GL}(F)$-submodule generated by y_1 is free (see [2], pp. 45–46). The element $y_2 = y_1^{p^m}$ also has this property, and $y_2 = \mathrm{Res}_F^C(y_2')$ for some $y_2' \in \mathrm{Ch}(C)$. Now let $y' = y_2' \mathrm{Res}_C^N(c_n(\rho))$, where ρ and n are as in the previous paragraph. Then y' is an element of $\mathrm{Ch}(C)$ whose restriction to an elementary abelian subgroup of C is non-zero only if that subgroup contains F. Moreover, the restriction to F of the representation ρ is invariant under $\mathrm{GL}(F)$, and so $y = \mathrm{Res}_F^C(y')$ generates a free $\mathbb{F}_p\mathrm{GL}(F)$-submodule of $\mathrm{Ch}(F)$. Now define $z_F \in H^*(G)$ by $z_F = \mathrm{Cor}_C^G(y')$. The Mackey formula gives

$$\mathrm{Res}_F^G(z_F) = \sum_{FgC} \mathrm{Cor}_{F \cap gCg^{-1}}^F c_g^* \mathrm{Res}_{g^{-1}Fg \cap C}^C(y').$$

Now $\mathrm{Res}_{F'}^C(y') = 0$ unless F' contains F, and so only those terms for which $g^{-1}Fg = F$ can be non-zero. Thus

$$\mathrm{Res}_F^G(z_F) = \sum_{g \in N/C} c_g^*(y),$$

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where the sum is over cosets of $C = C_G(F)$ in $N = N_G(F)$. Since y generates a free $\mathbb{F}_p\mathrm{GL}(F)$-summand of $\mathrm{Ch}(F)$, an automorphism f of F satisfies $f^*\mathrm{Res}_F^G(z_F) = \mathrm{Res}_F^G(z_F)$ if and only if f is equal to conjugation by some element of N. $\square$

Remark. The first part of this proof is very similar to Quillen's second proof of this statement, using the Evens norm map [17,11,5]. The second part is less similar however. Our argument is complicated by the weaker technique that we are using to construct elements, but is simplified by our use of Theorem 6.1 which means that we do not need to construct as many elements as are needed in [17].

Corollary 8.3 seems to be fairly well-known, although we have been unable to find it stated in the literature. Our first proof of Corollary 8.3 was essentially independent of the rest of this paper, but used a comparatively recent theorem of Carlson ([7], or theorem 10.2.1 of [11]): For G a p-group, with centre Z, the radical of $\ker(\mathrm{Res}_Z^G)$ is equal to the radical of the ideal generated by the images of Cor_H^G, where H ranges over all proper subgroups of G.

To prove Corollary 8.3 directly, note that if G_p is a Sylow p-subgroup of G, the transfer from $H^*(G_p)$ to $H^*(G)$ is surjective. Thus it suffices to consider the case when G is a p-group. Let G be a p-group, with centre Z. Using representations induced from Z up to G it may be shown that the image of $\mathrm{Ch}(G)$ in $H^*(Z)$ contains the subring of p^mth powers for sufficiently large m (as in the proof of Theorem 8.1). Thus if $y \in H^*(G)$, there exists m and $x_1 \in \mathrm{Ch}(G)$ such that $y_1 = y^{p^m} - x_1$ is in the kernel of Res_Z^G. By Carlson's theorem, there exists n, subgroups $H(1), \ldots, H(l)$ of G and $x_i' \in H^*(H(i))$ such that

$$y_2 = y_1^{p^n} = \sum_i \mathrm{Cor}_{H(i)}^G(x_i').$$

By induction on the order of G, there exists N such that, for each i, $(x_i')^{p^N} \in \overline{\mathrm{Ch}}(H(i))$. Noting that for x of degree $2i$,

$$\mathrm{Cor}_H^G(x^p) = \mathrm{Cor}_H^G(P^i x) = P^i \mathrm{Cor}_H^G(x) = \mathrm{Cor}_H^G(x)^p,$$

it follows that

$$y_2^{p^N} = \sum_i \mathrm{Cor}_{H(i)}^G((x_i'))^{p^N} = \sum_i \mathrm{Cor}_{H(i)}^G((x_i')^{p^N}) \in \overline{\mathrm{Ch}}(G).$$

9. A closure operation.

Definition. Let $\mathcal{C}$ be a category of elementary abelian subgroups of a group G. Define $\overline{\mathcal{C}}$, the closure of $\mathcal{C}$, to be the smallest subcategory of $\mathcal{C}_{\mathrm{reg}}$ such that:
  1. $\mathcal{C}$ is contained in $\overline{\mathcal{C}}$;
  2. if $f : E_1 \to E_2$ is in $\overline{\mathcal{C}}$, and $F_i \le E_i$ with $f(F_1) \le F_2$, then $f : F_1 \to F_2$ is in $\overline{\mathcal{C}}$;
  3. if $f : E_1 \to E_2$ is in $\overline{\mathcal{C}}$ and is an isomorphism of groups, then $f^{-1} : E_2 \to E_1$ is in $\overline{\mathcal{C}}$.

Say that $\mathcal{C}$ is closed if $\mathcal{C} = \overline{\mathcal{C}}$. Note that the categories $\mathcal{A}$, $\mathcal{A}'$, and $\mathcal{C}(R)$ for any $R \le H^*(G)$ are closed.

Proposition 9.1. *For any $\mathcal{C}$ containing the Quillen category $\mathcal{A}$, the category $\overline{\mathcal{C}}$ is the unique largest subcategory of $\mathcal{C}_{\mathrm{reg}}$ such that the natural map*

$$\mathrm{colim}_{\mathcal{C}} X_E(k) \to \mathrm{colim}_{\overline{\mathcal{C}}} X_E(k)$$

is a homeomorphism.

Proof. Let $\mathcal{D}$ be the subcategory of $\mathcal{C}_{\mathrm{reg}}$ whose morphisms $f : E \to F$ are those group homomorphisms that make the diagram

$$
\begin{array}{ccc}
X_E(k) & \xrightarrow{\ f\ } & X_F(k) \\
\downarrow{\iota} & & \downarrow{\iota} \\
\mathrm{colim}_{\mathcal{C}} X_E(k) & \xrightarrow{\ \mathrm{Id}\ } & \mathrm{colim}_{\mathcal{C}} X_E(k)
\end{array}
$$

commute. Then $\mathcal{D}$ has the property claimed, and it suffices to show that $\overline{\mathcal{C}} = \mathcal{D}$. Note also that $\overline{\mathcal{C}}$ is contained in $\mathcal{D}$ and that $\mathcal{D}$ is closed. Let $f \colon E_1 \to E_2$ be a morphism in $\mathcal{D}$. Since $f \colon E_1 \to E_2$ is in $\mathcal{D}$ (resp. in $\overline{\mathcal{C}}$) if and only if $f \colon E_1 \to f(E_1)$ is, it may be assumed that f is a group isomorphism. Let ϕ be an element of $X_{E_1}^+(k)$, *i.e.*, an element of $X_{E_1}(k)$ not contained in $X_F(k)$ for any proper subgroup F of E_1. Since $\mathrm{GL}(E_1)$ acts freely on $X_{E_1}^+(k)$, it follows that $f \colon E_1 \to E_2$ is uniquely determined by $\psi = f_*(\phi)$.

By definition of $\mathcal{D}$, ψ and ϕ have the same image in $\mathrm{colim}_{\mathcal{C}} X_E(k)$. Since $\mathcal{C}$ is skeletally finite (because it contains $\mathcal{A}$), there are chains $(F_0, \ldots, F_m)$ of objects of $\mathcal{C}$ and $(f_1, \ldots, f_m)$ of morphisms in $\mathcal{C}$, where

$$f_i \colon F_{i-1+\epsilon(i)} \to F_{i-\epsilon(i)}$$

for some $\epsilon(i) \in \{0, 1\}$, and $(\psi_0, \ldots, \psi_m)$, $\psi_i \in X_{F_i}(k)$, with

$$F_0 = E_1, \qquad F_m = E_2, \qquad \psi_0 = \phi, \qquad \psi_m = \psi, \qquad f_{i*}(\psi_{i-1+\epsilon(i)}) = \psi_{i-\epsilon(i)}.$$

Let F_i' be the unique subgroup of F_i such that $\psi_i \in X_{F_i'}^+(k)$. Then F_i' has the same rank as E_1 and f_i restricts to an isomorphism f_i' from $F_{i-1+\epsilon(i)}'$ to $F_{i-\epsilon(i)}'$. Letting $\delta(i) = 1 - 2\epsilon(i)$, $f_i'^{\delta(i)}$ is a morphism in $\overline{\mathcal{C}}$ from F_{i-1}' to F_i', and the composite

$$f' = f_m'^{\delta(m)} \circ \cdots \circ f_1'^{\delta(1)}$$

is a morphism in $\overline{\mathcal{C}}$ from E_1 to E_2 such that $f'(\phi) = \psi$. Hence $f' = f$, and f is a morphism in $\overline{\mathcal{C}}$ as claimed. $\square$

For any category $\mathcal{C}$ of elementary abelian subgroups of a group G, one may define a subring $R(\mathcal{C})$ of $H^*(G)$ as the inverse image of $\lim_{\mathcal{C}} H^*(E)$. This subring is large and is natural because $\lim_{\mathcal{C}} H^*(E)$ is.

Proposition 9.2. *For $\mathcal{C}$ any category of elementary abelian subgroups of G containing the Quillen category $\mathcal{A}$, $\mathcal{C}(R(\mathcal{C})) = \overline{\mathcal{C}}$.*

Proof. Clearly, $\mathcal{C}(R(\mathcal{C}))$ contains $\overline{\mathcal{C}}$, and is closed. Hence it suffices to show that the induced map of varieties is a homeomorphism. Quillen showed that the map from $H^*(G)$

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to $\lim_{\mathcal{A}} H^*(E)$ contains the subring of p^nth powers for some n (in fact this is equivalent to the injectivity of the map $\mathrm{colim}_{\mathcal{A}} X_E(k) \to X_G(k)$) [16]. Let $S = \lim_{\mathcal{C}} H^*(E)$, and note that if x is any element of S, the p^nth power of x is in the image of $R = R(\mathcal{C})$. It follows that the map $V_S(k) \to V_R(k)$ is a homeomorphism, and it suffices to show that the natural map

$$\mathrm{colim}_{\mathcal{C}} X_E(k) \to V_S(k)$$

is a homeomorphism. But this is a special case of lemma 8.11 of [16]. $\square$

The proposition shows that there is a sort of 'Galois correspondence' between large natural subrings of $H^*(G)$ and categories of elementary abelian subgroups of G.

10. Some other categories.

For each $n \geq 0$, define a category $\mathcal{A}^{(n)}$ of elementary abelian subgroups of a group G by declaring that the morphism $f\colon E \hookrightarrow F$ is in $\mathcal{A}^{(n)}$ if and only if for all $e_1, \ldots, e_n \in E$, there exists $g \in G$ such that $f(e_i) = g^{-1} e g$.

Note that $\mathcal{A}^{(0)}$ is the category $\mathcal{C}_{\mathrm{reg}}$ of Section 6, and $\mathcal{A}^{(1)}$ is the category $\mathcal{A}'$. For each n, $\mathcal{A}^{(n)} \supseteq \mathcal{A}^{(n+1)}$, and when n is greater than or equal to the p-rank of G, $\mathcal{A}^{(n)}$ is equal to Quillen's category $\mathcal{A}$. This suggests that $\mathcal{A}^{(\infty)}$ should be defined to be $\mathcal{A}$. Each of the categories $\mathcal{A}^{(n)}$ is closed in the sense of section 9, and the subrings $R^{(n)} = R(\mathcal{A}^{(n)})$ form a natural filtration of $H^*(G) = R^{(\infty)}$.

The categories $\mathcal{A}^{(n)}(G)$ are related to the generalized characters of G due to Hopkins, Kuhn and Ravenel [13] in the same way that the category $\mathcal{A}'$ is related to ordinary characters. It seems possible that there should be a description of the variety for the subring of elements of $H^*(G)$ coming from $E^0(BG)$, where E is a generalized cohomology theory to which Hopkins-Kuhn-Ravenel's work applies, in the same way that Chern classes are elements of $H^*(G)$ coming from $K^0(BG)$. We shall not make a precise conjecture, but shall give examples to show that the categories $\mathcal{A}^{(n)}$ can be distinct from each other.

Proposition 10.1. *For each $n \geq 0$ and each prime p, there is a p-group G for which $\mathcal{A}^{(n)}(G) \neq \mathcal{A}^{(n+1)}(G)$.*

Proof. For $n = 0$, the cyclic group of order p (for p odd), or the elementary abelian group of order four (for $p = 2$), will suffice. Hence we may assume that $n > 0$. Let C be a cyclic group of order p, let E be a faithful $\mathbb{F}_p C$-module of $\mathbb{F}_p$-dimension $n + 1$, and let $c \in \mathrm{GL}(E)$ represent the action on E of a generator for C. Now let Z be a vector space over $\mathbb{F}_p$ with basis $\{z_M\}$ indexed by the maximal $\mathbb{F}_p$-subspaces of E, so that Z has dimension $(p^{n+1} - 1)/(p - 1)$.

For each maximal subspace M of E, pick a linear map $\psi_M\colon E \to Z$, with kernel M and image generated by z_M. For each M, define $b_M \in \mathrm{GL}(E \oplus Z)$ by the equation

$$b_M(e, z) = (c(e), z + \psi_M(e)).$$

Let A be the subgroup of $\mathrm{GL}(E \oplus Z)$ generated by the b_M, and let G be the semidirect product $(E \oplus Z)\colon A$.

The subgroup Z is left invariant by A, so is central in G. Let ϕ be the homomorphism sending $A \leq \mathrm{GL}(E \oplus Z)$ to $\mathrm{GL}((E \oplus Z)/Z) \cong \mathrm{GL}(E)$, and let $B = \ker(\phi) \leq A$. Note that

elements of B act trivially on Z and on $(E \oplus Z)/Z$, and so B may be identified with a subgroup of the elementary abelian p-group $\mathrm{Hom}(E, Z)$.

We claim that the automorphism c of E is a morphism in $\mathcal{A}^{(n)}(G)$, but is not a morphism in $\mathcal{A}^{(n+1)}(G)$. If M is any rank-n subgroup of E, then the element $b_M \in G$ acts on M in the same way as c. On the other hand, if c were a morphism in $\mathcal{A}^{(n+1)}(G)$, there would have to be an element d of G, acting on $E \oplus Z$ as $d(e, z) = (c(e), z)$. But then, for any M, $d' = d^{-1}b_M$ would be an element of $(E \oplus Z) : B$ acting as $d'(e, z) = (e, z + \psi_M(e))$. To complete the proof, it suffices to show that there can be no such element d'.

Let R be the image of $\mathbb{F}_p C$ in the ring $\mathrm{End}(E)$. Since $\mathbb{F}_p C$ is a commutative local ring, it follows that R is too. In particular, the non-units in R form an ideal. Fix M, a maximal subgroup of E. The group $B \leq A \leq \mathrm{GL}(E)$ is generated as a normal subgroup by the elements b_M^p, and $b_M^{-1}b_N$, where N ranges over all other maximal subgroups of E. The action of these generators on $E \oplus Z$ is given by:

$$b_M^p(e, z) = \left(c^p(e), z + \sum_{i=0}^{p-1} \psi_M(c^i(e)) \right) = (e, \psi_M(\bar{r}e) + z),$$

where $\bar{r}$ is the image of $\bar{c} = \sum_{i=0}^{p-1} c^i$ in $R = \mathrm{End}(E)$, and

$$b_M^{-1}b_N(e, z) = (e, z + \psi_N(e) - \psi_M(e)).$$

We therefore have to show that the element d' described above does not lie in the subgroup of $\mathrm{GL}(E \oplus Z)$ generated by

$$(e, z) \mapsto (e, z + \psi_M(\bar{r}e)) \qquad \text{and} \qquad (e, z) \mapsto (e, z + \psi_N(c^i e) - \psi_M(c^i e)),$$

for all $N \neq M$ and $0 \leq i \leq p - 1$.

If $d' \colon (e, z) \mapsto (e, z + \psi_M(e))$ is in the subgroup B, there are $\mu, \lambda_N \in R \leq \mathrm{End}(E)$ such that

$$\mathrm{Im}\lambda_N \subseteq \ker(\psi_N) = N \quad \text{for all } N \neq M, \text{ and}$$

$$\mathrm{Im}\left(1 - \mu\bar{r} + \sum_{N \neq M} \lambda_N \right) \subseteq \ker(\psi_M) = M.$$

From the first family of equations, it follows that each λ_N is a non-unit in R, and of course $\bar{r}$ is a non-unit. But then $1 - \mu\bar{r} + \sum \lambda_N$ must be a unit in R, contradicting the second equation. $\square$

Remark. We exhibit p-group examples above because p-groups control the behaviour of mod-p cohomology, and hence p-group examples tend to be harder to find. Similar examples can be constructed for other cyclic groups C. One interesting example is the case when $C = \mathrm{GL}_1(\mathbb{F}_q)$ for $q = p^{n+1} > 2$ and E is the additive group of $\mathbb{F}_q$. In this case $\mathbb{F}_p C$ is not a local ring, but its image in $\mathrm{End}(E)$ is isomorphic to $\mathbb{F}_q$, and $\bar{r} = 0$, so the argument used above still applies.

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